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Residual Problems in ALADIN-NH Dynamical Core

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1 Introduction

Recent developments in ALADIN-NH (PC scheme, prognostic variable d_4, diagnostic BBC, `SITRA`) resulted in efficient and robust dynamical core, which can be assumed satisfactory in wide range of regimes. Configuration sl2tl + d_4 + `SITRA` was found so stable that it does not require PC iterations (SI correction is sufficient for stability). However, currently there are known two remaining problems concerning the new dynamical core:

1. Semi-lagrangian chimney problem was solved either by advection of w (C. Smith, [3]) or by diagnostic BBC (P. Smolíková, [6]). But it was detected in [1] that using horizontal diffusion can create chimney even for eulerian advection scheme.

2. As was shown by J. Vívoda, numerical solution in André Robert's bubble test becomes deteriorated when advection of d variable is used. The problem can be avoided by advection of w developed by C. Smith (variable d_0) and generalized by J. Vívoda (variables d_3 and d_4, but only for non-extrapolating PC scheme).

Work described in this report was partly a continuation of my previous stay in Prague (May–June 2003), partly a reaction to some recent results. Main target was to answer following questions:

- What is the mechanism of eulerian chimneys? Is there some connection with semi-lagrangian ones?

- Does bubble test indicate some fundamental problems connected to d-type prognostic variables?

- Is advection of w unavoidable?

2 Chimney problem and BBC formulation

In my previous report [1] it was stated that chimneys are connected to non-linear regimes, which makes their understanding difficult. This statement turned out to be false, since in [6] semi-lagrangian chimney was observed also for linear potential flow. Question arose whether horizontal diffusion can produce eulerian chimney in this regime. R. Brožková confirmed that the answer is positive. As a next step she proposed to test if horizontal diffusion will restore chimney for semi-lagrangian scheme with advection of w or diagnostic BBC. This was checked in subsequent experiments.

2.1 Setup of experiments

Experiments were done using 2D vertical plane model. Non-linear non-hydrostatic (NLNH) orographic flow was used. This is the regime where eulerian chimney was observed for the first time:

- Initial state:
  - temperature profile with constant Brunt-Väisälä frequency $N = 0.01\,\mathrm{s}^{-1}$ up to tropopause at height 21 km, isothermal above tropopause
  - sea level temperature 293 K
  - tropopause temperature 133 K
  - constant wind profile with $V = 10\,\mathrm{ms}^{-1}$
  - sea level pressure 101 325 Pa
- Orography: Bell shaped mountain.

  height: $h = 1000\,\mathrm{m}$
  half-width: $a = 1000\,\mathrm{m}$
- Dimensionless flow parameters:

  $$C_L = \frac{Nh}{V} = 1.0 \quad (C_L \ll 1 \Rightarrow \text{linear flow})$$

  $$C_H = \frac{V}{Na} = 1.0 \quad (C_H \ll 1 \Rightarrow \text{hydrostatic flow})$$
- Geometry:

Δx	[m]	200	($a = 5\Delta x$)
Δz	[m]	≈300	(regular z-levels)
`NDGUX`		128	(C+I zone)
`NDGL`		128	(no E zone)
`NBZONG`		14	(I zone)
`NSMAX`		42	(quadratic grid)
`NFLEVG`		100	(30 levels above tropopause)

- Vertical coordinate: σ
- Coupling files: Identical with initial file (time constant LBC).

- Common integration settings:

t_{STOP}	[s]	5000
NPDVAR		2
NVDVAR		3
SIPR	[Pa]	90000.
REPONBT	[m]	20000.
REPONTP	[m]	29500.
HDIRT	[s]	0.
HDIRVOR	[s]	0.
VESL		0.
XIDT		0.

- Scheme dependent integration settings:

		euler	sl2tl
TSTEP*	[s]	1.	10.
REPONTAU**	[s]	100.	50.
RCMSLP0		0.	1.
SITR	[K]	220.	300.
SITRA	[K]	220.	50.
LPC		OLD	FULL NESC
NSITER		1	3

(∗) CFL criterion would enable to use timestep $\Delta t = 2.5\,\text{s}$ with eulerian scheme. In order to get reference solution, timestep was reduced to 1.0 s.

(∗∗) Due to the bug in SUPONG, sponge applied in 3 time level scheme is two times stronger than in 2 time level scheme (using the same absorption timescale REPONTAU). That is the reason why different value of REPONTAU was used with 2 time level scheme.

- Experiment dependent integration settings:

figure	scheme	HDIRDIV HDIRVD [s]	remark
1	euler	0.	eulerian
2		5.	reference
3	sl2tl	0.	semi-lagrangian
4		5.	reference
5	sl2tl + LGWADV	0.	advection of w
6		5.	
7	sl2tl + LRDBBC	0.	diagnostic BBC
8		5.	

When horizontal diffusion was used, it was applied only on horizontal divergence D and vertical divergence d. Temperature T and vorticity ξ were not diffused (vorticity in 2D model is zero).

2.2 Experimental results

Figures 1–8 show integration results after 5000 s. Basic flow is from left to right. Field of vertical velocity w is displayed. It can be observed that:

- Reference eulerian solution is a bit noisy (fig. 1). Turning on horizontal diffusion removes the noise but creates chimney (fig. 2).
- Reference semi-lagrangian solution is less noisy, but it contains chimney (fig. 3). Turning on horizontal diffusion amplifies the chimney (fig. 4).
- Advection of w (fig. 5) or diagnostic BBC (fig. 7) removes semi-lagrangian chimney. Obtained solutions are slightly different, especially for the first maximum behind the mountain.
- Turning on horizontal diffusion restores chimney in semi-lagrangian mode both for advection of w and diagnostic BBC (fig. 6, 8).

2.3 Theoretical analysis and proposed solutions

Experimental results revealed that horizontal diffusion applied on variables (D, d) creates a chimney, independently on used advection scheme (it would therefore be more precise to speak about *diffusive* chimney rather than *eulerian* chimney). This indicates that diffusive chimney might not be related to model discretization. Taking into account the fact that diffusive chimney can be observed also in linear regimes lead to the attempt examine the problem using linearized continuous equations.

Analysis based on linear Long model was performed by P. Bénard and it was a great success. It is outlined in appendix A. Main result can be illustrated on equation (4) from [1], which prescribes BBC for the term $\frac{\partial \tilde{p}}{\partial \pi}$ consistently with dynamical equations:

$$\left(\frac{\partial \tilde{p}}{\partial \pi}\right)_S = \frac{\left[-\frac{RT}{p}\nabla p - \nabla\phi + \boldsymbol{\mathcal{V}}\right]_S \cdot \nabla\phi_S + J_S - g\mathcal{W}_S}{g^2 + (\nabla\phi_S)^2} \tag{1}$$

$$\left(J_S = \frac{\partial^2 \phi_S}{\partial x^2}{u_S}^2 + 2\frac{\partial^2 \phi_S}{\partial x \partial y}u_S v_S + \frac{\partial^2 \phi_S}{\partial y^2}{v_S}^2\right)$$

Occurence of source terms $\boldsymbol{\mathcal{V}}$ and $\mathcal{W}$ in formula (1) is crucial. They should consist of three contributions:

1. Coriolis acceleration
2. horizontal diffusion
3. physical tendencies

However, in the model only Coriolis acceleration is incorporated into BBC. Contributions from horizontal diffusion and physical tendencies are missing. This was not recognized in [1]. Analysis of Long model showed that diffusive chimney is caused by ignoring second contribution. Further problems can be expected for diabatic model, where also third contribution can be non-zero.

Incorporating physical tendencies into BBC should not be a problem, especially when the physics is computed before dynamics. Situation is more complicated for horizontal diffusion, since it is computed in spectral space at the end of timestep, while BBC is evaluated in gridpoint space during computation of RHS. There are several possible solutions, using different degrees of approximation:

- Exact solution would require transformation of fields $\mathbf{v}_L$, $w_{\tilde{L}}$ into spectral space, computation of diffusion terms[1] $-K_D\nabla^4\mathbf{v}_L$, $-K_d\nabla^4 w_{\tilde{L}}$ and their transformation back to gridpoint space. This procedure would have to be repeated in each timestep.
- Analysis of linear Long model suggests that dominant contribution of horizontal difussion to the BBC comes from the term $-K_d\nabla^4 w_{\tilde{L}}$, which can be approximated as $-K_d\,\mathbf{v}_L\cdot\nabla\left(\nabla^4\phi_{\tilde{L}}\right)$. Main advantage over exact solution is due to the fact that surface geopotential $\phi_{\tilde{L}}$ does not depend on time. It would therefore be enough to compute term $\nabla\left(\nabla^4\phi_{\tilde{L}}\right)$ at the beginning of integration, transform it to gridpoint space and store it.
- Newly developed SLHD (Semi-Lagrangian Horizontal Diffusion) could simplify the things, because it is computed in gridpoint space.

Remark:

In the model formula (1) is not used directly, because some quantities are not available at surface level $\tilde{L}$. It is therefore modified using assumption $\mathbf{v}_{\tilde{L}} = \mathbf{v}_L$:

$$g^2\left(\frac{\partial\tilde{p}}{\partial\pi}\right)_{\tilde{L}} = \left[-\frac{RT}{p}\nabla p - \left(\frac{\partial\tilde{p}}{\partial\pi}+1\right)\nabla\phi + \boldsymbol{\mathcal{V}}\right]_L\cdot\nabla\phi_{\tilde{L}} + J_L - g\mathcal{W}_{\tilde{L}} \tag{2}$$

Main problem with formula (2) is that it contains term $\left(\frac{\partial\tilde{p}}{\partial\pi}\right)_L$, which requires half level values $\tilde{p}_{\tilde{L}}$ and $\tilde{p}_{\tilde{L}-1}$. In current code surface value is determined by extrapolation as $\tilde{p}_{\tilde{L}} = \tilde{p}_L$, which corresponds to $\left(\frac{\partial\tilde{p}}{\partial\pi}\right)_{\tilde{L}} = 0$. This may be in contradiction with expression (2). More consistent approach was studied in [1], but surprisingly there was almost no impact on model results.[2]

2.4 Conclusions

Main conclusions were drawn by P. Bénard. They can be summarized into these points:

- Chimney problem is always caused by inconsistency between dynamical equations and BBC.
- Diffusive chimney is caused by inconsistency between diffused d-equation and BBC obtained from undiffused w-equation.
- Semi-lagrangian chimney is caused by numerical inconsistency between dynamical equations discretized in semi-lagrangian manner and BBC diagnosed in eulerian way.

[1] In model, horizontal diffusion is applied to vorticity ξ and horizontal divergence D via source terms $-K_\xi\nabla^4\xi$, resp. $-K_D\nabla^4 D$. When $K_\xi \neq K_D$, equivalent source term for velocity $\mathbf{v}$ does not take the simple form $-K_D\nabla^4\mathbf{v}$, but is given by more cumbersome formula:

$$u:\quad -(K_D\partial_x^2 + K_\xi\partial_y^2)\nabla^2 u - (K_D - K_\xi)\partial_{xy}^2\nabla^2 v$$
$$v:\quad -(K_\xi\partial_x^2 + K_D\partial_y^2)\nabla^2 v - (K_D - K_\xi)\partial_{xy}^2\nabla^2 u$$

[2] Remark for Alena: Yes, I have used the new master.

3 Bubble tests

During tests of PC scheme J. Vívoda noticed patological behaviour in bubble experiments when advection of d variable was used. It could be suppressed by advecting w. Even before these tests it was pointed out by C. Smith that d-type prognostic variables have several drawbacks compared to w. They are listed in [3]. Two of them might be responsible for problems in bubble experiments:

- bigger discretization error due to more complicated RHS of d-equation
- bad error propagation characteristics due to so called Z-term in d-equation

Mentioned drawbacks can be explained on prognostic equations for w and d_3. They can be found in [5] and rearranged into the form:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = g\frac{\partial \tilde{p}}{\partial \pi} + \mathcal{W} \tag{3}$$

$$\frac{\mathrm{d}d_3}{\mathrm{d}t} = g^2\frac{\partial}{\partial \phi}\frac{\partial \tilde{p}}{\partial \pi} + g\frac{\partial \mathcal{W}}{\partial \phi} - d_3\left[d_3 \underbrace{-\frac{\partial \mathbf{v}}{\partial \phi}\cdot\nabla\phi}_{X}\right] \underbrace{-\frac{\partial \mathbf{v}}{\partial \phi}\cdot\nabla(gw)}_{Z} \tag{4}$$

$$d_3 \equiv \frac{\partial}{\partial \phi}(gw) = -\frac{p}{mRT}\frac{\partial}{\partial \eta}(gw)$$

$$gw = gw_S + \int_\eta^1 \frac{mRT}{p} d_3\,\mathrm{d}\eta \tag{5}$$

RHS of equation (4) contains several problematic terms for discretization. First term evaluated at lowest full level requires BBC for derivative $\frac{\partial \tilde{p}}{\partial \pi}$. As was mentioned in section 2, dynamically inconsistent BBC for this term is responsible for chimney formation.

Another problematic terms are X and Z. Their discretization at full levels requires use of vertical averaging. Approach used in model is not optimal. C. Smith proposed alternative approach in [3], which has generally smaller leading error term for irregularly spaced levels. Testing of this approach was not finished in [1].

One more problem connected to Z-term is the fact that it contains gradient of w. Velocity w must be diagnosed using formula (5). Undesirable feature of vertical integral occuring in (5) is that it immediately propagates errors contained in fields π_S, p, T, d_3 upward.

All mentioned problems could be avoided by using w as prognostic variable. But there are two main obstacles to do this:

1. Stability of SI scheme for prognostic variable w is insufficient. It is equivalent to stability of variable d_0.
2. Velocity w is half level quantity. Using it as prognostic variable would require search of extra origin points for final points located at half levels. This is against ALADIN philosophy where effort was made to have all prognostic variables as full level quantities.

Considering first point, advection of w can be viewed as a compromise. Variable w is used in gridpoint computations (simple RHS, no Z-term), variable d_3 or d_4 during SI correction (sufficient stability). As for second point, extra origin points are still needed. That is why different solution is searched.

Warning:

Dry atmosphere ($R = R_d$) was assumed in equations (3)–(5).

3.1 Setup of experiments

Experiments were done using 2D vertical plane model, following André Robert's warm and cold bubble test described in [2]:

- Initial state:
  - neutral background profile (constant potential temperature $\theta_0 = 300\,\mathrm{K}$) superposed with bubble perturbations of the form:

$$\theta' = \begin{cases} A & ; \quad r \leq a \\ A \exp\left[-\frac{(r-a)^2}{s^2}\right]; & r > a \end{cases}$$

$$r = (y - y_0)^2 + (z - z_0)^2$$

	A [K]	a [m]	s [m]	y_0 [m]	z_0 [m]
warm bubble	0.50	150	50	500	300
cold bubble	-0.15	0	50	560	640

  - resting, vertically balanced state*
  - sea level pressure 101 325 Pa

  (∗) In article [2] initial state was balanced horizontally. There was no horizontal pressure gradient force, only buoyant force in the vertical. For ALADIN-NH it was simpler to prepare vertically balanced initial state (no buoyant force) with some resulting horizontal pressure gradient force. Since the potential temperature perturbation θ' is small, there is only slight difference between the two initial states. Moreover, adjustment process takes part during early stages of integration, radiating away small initial imbalance.

- Orography: Flat.
- Geometry:

Δx	[m]	10	
Δz	[m]	≈10	(regular z-levels)
`NDGUX`		`120/100`	(C+I zone)
`NDGL`		`120/100`	(no E zone)
`NBZONG`		`0`	(no I zone)
`NSMAX`		`39/32`	(quadratic grid)
`NFLEVG`*		`120/100`	

  (∗) In ALADIN-NH elastic TBC is used. It has the form $p_T = 0$ (more general case $p_T = const$ is not coded). In order to prevent huge jump in resolution and pressure at the top, additional 30 levels were added to the top of model domain. Their spacing Δz increases by factor 1.2 between adjacent layers. They are not shown in the plots.

- Vertical coordinate: η

$$A = \sigma(1-w) \qquad w = (3-2\sigma)\sigma^2$$
$$B = \sigma w \qquad \sigma = A + B$$

- Coupling files: None (periodic domain).
- Common integration settings:

t_{STOP}	[s]	600
NPDVAR		2
NVDVAR		3
SIPR	[Pa]	90000.
VESL		0.
XIDT		0.

- Scheme dependent integration settings:

		euler	sl2tl
SITR	[K]	250.	350.
SITRA	[K]	250.	100.
LPC			FULL NESC
NSITER		0	1

- Experiment dependent integration settings:

figure	scheme	Δt [s]	HDIRDIV HDIRVD [s]	HDIRVOR HDIRT [s]	L_y, L_z [km]	reverted
9–12	sl2tl + LGWADV	5.0	0.	0.	1.2	no
13–16		5.0	0.	0.	1.2	yes
17–20		5.0	0.	0.	1.0	yes
21	sl2tl	5.0	0.	0.	1.2	no
22		1.0	0.	0.	1.2	no
23		0.2	0.	0.	1.2	no
24		5.0	0.	0.	1.2	yes
25	sl2tl + LRDBBC	5.0	0.	0.	1.2	no
26		1.0	0.	0.	1.2	no
27		0.2	0.	0.	1.2	no
28		5.0	0.	0.	1.2	yes
29	euler	1.0	0.	0.	1.2	no
30		1.0	5.	25.	1.2	no
31		0.2	5.	25.	1.2	no
32		1.0	5.	25.	1.2	yes

3.2 Experimental results

Warm and cold bubble test in [2] was performed on domain 1×1 km with rigid lid TBC. During integration rising bubble started to interact with upper boundary and distort. In ALADIN-NH elastic TBC is implemented. That is why integration domain was enlarged to 1.2×1.2 km, so that rising bubble is not influenced by upper boundary during first 600 s.

On figures 9–32 field of potential temperature perturbation $\theta' = \theta - \theta_0$ is displayed. Following observations can be made:

- Results from sl2tl scheme with advection of w are shown on fig. 9–12. They can be compared with fig. 8a–8d in [2]. Agreement is very good. There is a weak noise present in ALADIN-NH fields (fig. 11, 12). Top of warm bubble on fig. 12 is not flat as in the article because of different upper boundary.

- When Boussinesq approximation is applicable, reverted bubble test should give the same results as direct test. Reverted test can be prepared by vertical mirroring of initial state with change of sign for vertical velocity w and potential temperature perturbation θ'. After integration this procedure is repeated in order to get results comparable with direct test. See appendix B for more details.

  Experiment confirms that results of reverted bubble test (fig. 13–16) are very close to those of direct test (fig. 9–12). Slight difference can be seen at lower part of warm bubble (compare fig. 10 and 14). See remark at the end of appendix B for possible explanations.

- Reverted bubble test enables to simulate rigid lid TBC even with ALADIN-NH, because upper and lower boundaries are interchanged. This trick makes it possible to reproduce results from [2] obtained on 1×1 km domain (fig. 17–20). Again, weak noise can be seen in ALADIN-NH solution (fig. 19, 20).

- Problems occur when sl2tl scheme with advection of d_3 is used. Solution is strongly distorted (fig. 21). Distortion does not disappear when shorter timesteps are used (fig. 22, 23). Reverted test gives better results, which are however not consistent with direct test (fig. 24).

- Using sl2tl scheme with diagnostic BBC does not improve the things (fig. 25–28). Results are similar as in previous case, especially for short timesteps.

- Eulerian scheme gives very noisy fields (fig. 29). Horizontal diffusion must be applied in order to get acceptable results (fig. 30–32). Eulerian response is a bit different from semi-lagrangian one (compare fig. 12 and 30), but the solution is probably correct. There is no sensitivity shorter timestep (fig. 31). Reverted test is in good agreement with direct test (fig. 32).

In all experiments noise was generated at the top of model domain. It contaminated several uppermost levels during integration (they are not displayed in the plots). Noise was strongest for eulerian scheme, then for problematic semi-lagrangian configurations.

Code with alternative discretization of terms X and Z was debugged and tested. There was hardly any impact on bubble experiments.

3.3 Conclusions

- Model discretization of terms X and Z is not responsible for spurious behaviour in bubble experiments. Alternative discretization had almost no impact on results. This is not surprising, since in bubble test regular z-levels were used and in such case both discretizations should be equivalent.

- Problems might originate from error propagation characteristics due to Z-term. This is supported by the fact that advection of w suppresses spurious behaviour as well as by asymmetry between direct and reverted bubble test.

- Results from eulerian integrations seem to be in contradicton with previous point. They should suffer from the same defficiencies as semi-lagrangian ones, because Z-term is still present in d-equation. But this is not the case. It is therefore probable that semi-lagrangian interpolations play some role in deteriorating solution. It seems as if errors in diagnosing w were compensating for eulerian scheme.

4 Info section

4.1 Unfinished work

- Dynamically consistent BBC for term $\frac{\partial \tilde{p}}{\partial \pi}$ was not coded. Experimental work will be needed to decide which approximations in BBC formulation are acceptable.
- Question whether advection of w can be avoided remained open. Patological behaviour in bubble experiments was not fully understood yet. Problems may arise from error propagation characteristics. More research in this direction will be needed, employing simplified models like the one described in [4].

4.2 Code info

All work was based on cycle `25t2`. Several versions of the code were used:

`30` = reference version + modifications from J. Vívoda
(bugfix, `SITRA`)

`31` = `30` + modifications from J. Vívoda
(combination `LGWADV` + `LPC_FULL` + `LPC_NESC` enabled for all d variables)

`33` = `31` + alternative discretization of terms X and Z

`40` = reference version + modifications from P. Smolíková
(d_4 bugfix, `SITRA`, `LRDBBC`)

Modified sources (`voodoo`):

```
~mma157/utemp/cycle_25t2/mod_30_ald/
                         mod_30_arp/
                         mod_31d30_ald/
                         mod_31d30_arp/
                         mod_33d31_ald/
                         mod_33d31_arp/
                         mod_40_ald/
                         mod_40_arp/
```

Sources + dependencies for compilation (`lambda`):

```
~mma157/work/cycle_25t2/dep_30_ald/
                        dep_30_arp/
                        dep_31_ald/
                        dep_31_arp/
                        dep_33_ald/
                        dep_33_arp/
                        dep_40_ald/
                        dep_40_arp/
```

Loading scripts (`lambda`):

```
~mma157/work/cycle_25t2/load/load_30_sx6
                             load_31_sx6
                             load_33_sx6
                             load_40_sx6
```

Executables (`archiv`):

```
~mma157/bin/master_al25t2_30_sx6
             master_al25t2_31_sx6
             master_al25t2_33_sx6
             master_al25t2_40_sx6
```

Integration scripts (`sx6`):

```
~mma157/m2d/exp/script_04/
```

5 Final remark

> "Consistent lie is better than inconsistent truth."
>
> anonymous

Appendix

A Long solution

This section briefly outlines analysis of diffusive chimney performed by P. Bénard. It is done in linear framework using vertical σ coordinate and restricted to $x\sigma$ plane. Most notations are taken from ALADIN-NH documentation.

Model prognostic variables are horizontal divergence D, vertical divergence d (resp. vertical velocity w), thermodynamic temperature T, NH pressure departure $\mathcal{P}$ and logarithm of MSL pressure Q which replaces variable $q = \ln \pi_S$ used in ALADIN-NH:

$$\begin{aligned}
D &\equiv \frac{\partial u}{\partial x} \\
d &\equiv -g\frac{1+\mathcal{P}}{RT}\sigma\frac{\partial w}{\partial \sigma} \\
\mathcal{P} &\equiv \frac{p-\pi}{\pi} \qquad (p \text{ – true pressure, } \pi \text{ – mass coordinate}) \\
Q &\equiv q + \frac{\phi_S}{RT^*} = \ln \pi_S + \frac{\phi_S}{RT^*}
\end{aligned}$$

Background state for linearization is taken isothermal (constant temperature T^*), hydrostatically balanced, with flat orography $\phi_S^* = 0$ and uniform background wind u^*. Background state is disturbed by introducing small orography perturbation ϕ_S and the response of linearized system is studied. Stationary solutions are seeked. Perturbations of prognostic variables T, Q are denoted T', Q'. For remaining variables primes are omitted, since corresponding background values are zero.

Prognostic equation for perturbation variable X can be symbolically written in the form:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mathrm{RHS}_X$$

Since stationary solutions of linearized system are examined, LHS of this equation can be simplified:

$$u^*\nabla X = \mathrm{RHS}_X \qquad (\nabla \equiv \partial_x) \tag{6}$$

When horizontal diffusion is applied on variable X, equation (6) becomes:

$$\begin{aligned}
u^*\nabla X &= \mathrm{RHS}_X - K_X\nabla^4 X \\
u^*\nabla X + K_X\nabla^4 X &= \mathrm{RHS}_X \\
(u^* + K_X\nabla^3)\nabla X &= \mathrm{RHS}_X \\
\bar{u}_X\nabla X &= \mathrm{RHS}_X \qquad (\bar{u}_X \equiv u^* + K_X\nabla^3)
\end{aligned} \tag{7}$$

Replacing u^* by operator $\bar{u}_X$ is a very elegant way how to introduce horizontal diffusion into equations. When operator $\bar{u}_X$ is applied on function e^{ikx}, eigenvalue $\hat{u}_X(k)$ can be useful:

$$\begin{aligned}
\bar{u}_X\mathrm{e}^{ikx} &= \hat{u}_X(k)\mathrm{e}^{ikx} \\
\hat{u}_X(k) &= u^* - ik^3K_X
\end{aligned} \tag{8}$$

In subsequent text following vertical operators are used (function f must be defined for $\sigma \in [0,1]$):

$$(\mathcal{G}f)(\sigma) \equiv \int_\sigma^1 \frac{f(\sigma')}{\sigma'}\,\mathrm{d}\sigma'$$

$$
\begin{aligned}
(\mathcal{S}f)(\sigma) &\equiv \frac{1}{\sigma}\int_0^\sigma f(\sigma')\,\mathrm{d}\sigma' \\
(\mathcal{N}f)(\sigma) &\equiv \int_0^1 f(\sigma')\,\mathrm{d}\sigma' \\
\mathcal{I}f &\equiv f \qquad (\mathcal{I}\text{ is identity}) \\
\tilde{\partial} &\equiv \sigma\frac{\partial}{\partial\sigma} \\
\mathcal{L} &\equiv \tilde{\partial}(\tilde{\partial}+\mathcal{I})
\end{aligned}
$$

There are many relations between these operators, most useful ones are listed below. They are handy e.g. during derivation of structure equations:

$$
\begin{aligned}
\mathcal{G}\mathcal{S} &= \mathcal{G}+\mathcal{S}-\mathcal{N} \\
\mathcal{S}\mathcal{G} &= \mathcal{G}+\mathcal{S} \\
\tilde{\partial}\mathcal{S} &= \mathcal{I}-\mathcal{S} \\
\tilde{\partial}\mathcal{G} &= -\mathcal{I} \\
\mathcal{L}\mathcal{S} &= \tilde{\partial} \\
\mathcal{L}\mathcal{G} &= -(\tilde{\partial}+\mathcal{I}) \\
\mathcal{L}\mathcal{S}\mathcal{G} &= -\mathcal{I}
\end{aligned}
$$

A.1 Analysis for prognostic variable d

In this case, with horizontal diffusion used only for variables D and d, linearized prognostic equations describing stationary flow have the form:

$$\bar{u}_D\nabla D = \Delta[-R\mathcal{G}T' + RT^*(\mathcal{G}-\mathcal{I})\mathcal{P} - RT^*Q'] \tag{9}$$

$$\bar{u}_d\nabla d = -\frac{g^2}{RT^*}\mathcal{L}\mathcal{P} \tag{10}$$

$$u^*\nabla T' = -\frac{RT^*}{c_v}(D+d) \tag{11}$$

$$u^*\nabla\mathcal{P} = \mathcal{S}D - \frac{c_p}{c_v}(D+d) \tag{12}$$

$$u^*\nabla Q' = -\mathcal{N}D + \frac{1}{RT^*}u^*\nabla\phi_S \tag{13}$$

Structure equation can be obtained from system (9)–(13) by eliminating all variables except d:

$$\nabla\left[-\frac{1}{c^2}\bar{u}_D\bar{u}_d u^{*2}\Delta + u^*\left(\bar{u}_d\Delta + \bar{u}_D\frac{\mathcal{L}}{H^2}\right) + N^2\right] d = 0 \tag{14}$$

$$c \equiv \frac{c_p}{c_v}RT^* \qquad H \equiv \frac{RT^*}{g} \qquad N^2 \equiv \frac{g^2}{c_pT^*}$$

General solution of equation (14) can be written as a superposition of particular solutions having form:

$$d = \hat{d}\,\mathrm{e}^{ikx}\sigma^{i\nu-\frac{1}{2}} \tag{15}$$

$$k\in\mathbb{R} \qquad \hat{d},\nu\in\mathbb{C}$$

Inserting (15) into structure equation (14) leads to dispersion formula giving relation between dimensionless vertical wavenumber ν and horizontal wavenumber k:

$$\frac{\nu^2}{H^2} = \frac{N^2}{u^*\hat{u}_D} - k^2\frac{\hat{u}_d}{\hat{u}_D}\left(1-\frac{u^*\hat{u}_D}{c^2}\right) - \frac{1}{4H^2} \tag{16}$$

Now it is necessary to determine complex amplitude $\hat{d}$ corresponding to monochromatic orographic forcing $\phi_S = \hat{\phi}_S \, e^{ikx}$. This can be done using free slip BBC together with prognostic equation for w evaluated at surface:

$$gw_S = u^* \nabla \phi_S \qquad \bar{u}_w \nabla w_S = [g(1+\tilde{\partial})\mathcal{P}]_S$$
$$\Downarrow$$
$$u^* \bar{u}_w \Delta \phi_S = [g^2(1+\tilde{\partial})\mathcal{P}]_S$$

Inserting expressions $\phi_S = \hat{\phi}_S \, e^{ikx}$ and $\mathcal{P} = \hat{\mathcal{P}} \, e^{ikx} \sigma^{i\nu - \frac{1}{2}}$ into the last equation gives formula for amplitude $\hat{\mathcal{P}}$:

$$\hat{\mathcal{P}} = -\frac{k^2 u^* \hat{u}_w}{g^2 \left(i\nu + \dfrac{1}{2} \right)} \hat{\phi}_S \tag{17}$$

Polarization relation between amplitudes $\hat{d}$ and $\hat{\mathcal{P}}$ can be obtained by analogical way from equation (10):

$$ik\hat{u}_d \hat{d} = \frac{g^2}{RT^*} \left(\nu^2 + \frac{1}{4} \right) \hat{\mathcal{P}} \tag{18}$$

Elimination of $\hat{\mathcal{P}}$ from equations (17) and (18) gives:

$$\hat{d} = -\frac{1}{RT^*} \left(i\nu - \frac{1}{2} \right) ik \frac{u^* \hat{u}_w}{\hat{u}_d} \hat{\phi}_S \tag{19}$$

Final step is to express vertical divergence field $d(x, \sigma)$ using (19) and convert it into vertical velocity field $w(x, \sigma)$. This can be achieved employing linearized diagnostic formula:

$$gw(x, \sigma) = gw_S(x) + RT^* \int_\sigma^1 \frac{d(x, \sigma')}{\sigma'} \, d\sigma'$$

Together with free slip BBC this leads to the result:

$$gw(x, \sigma) = iku^* \hat{\phi}_S \, e^{ikx} \left[\left(1 - \frac{\hat{u}_w}{\hat{u}_d} \right) + \frac{\hat{u}_w}{\hat{u}_d} \sigma^{i\nu - \frac{1}{2}} \right] \tag{20}$$

Formula (20) can be further simplified realizing that horizontal diffusion for variables d and w must be the same, i.e. $\hat{u}_d = \hat{u}_w$. This comes from the fact that prognostic equation for d is obtained by vertical differentiation of prognostic equation for w. So the solution (20) can be written as:

$$gw(x, \sigma) = iku^* \hat{\phi}_S \, e^{ikx} \sigma^{i\nu - \frac{1}{2}} \tag{21}$$

In current model formulation, however, horizontal diffusion for w is ignored when deriving BBC for $\mathcal{P}$. In other words $K_w = 0$ and thus $\hat{u}_w = u^*$. When inserted into equation (20) this gives:

$$gw(x, \sigma) = iku^* \hat{\phi}_S \, e^{ikx} \left[\Big(\underbrace{1 - \frac{u^*}{\hat{u}_d}}_{\text{chimney}} \Big) + \frac{u^*}{\hat{u}_d} \sigma^{i\nu - \frac{1}{2}} \right] \tag{22}$$

It can be seen immediately that when horizontal diffusion for variable d is turned on ($K_d > 0$, $\hat{u}_d \neq u^*$), spurious pattern without vertical structure appears in w field (22). This is famous diffusive chimney. Remaining part of the solution is further distorted by factor $\frac{u^*}{\hat{u}_d}$, as is clear from comparison with (21).

A.2 Analysis for prognostic variable w

In this case, with horizontal diffusion used only for variables D and w, linearized prognostic equations describing stationary flow have the form:

$$\bar{u}_D \nabla D = \Delta[-R\mathcal{G}T' + RT^*(\mathcal{G} - \mathcal{I})\mathcal{P} - RT^*Q'] \tag{23}$$

$$\bar{u}_w \nabla w = g(\tilde{\partial} + \mathcal{I})\mathcal{P} \tag{24}$$

$$u^* \nabla T' = -\frac{RT^*}{c_v}\left(D - \frac{1}{H}\tilde{\partial} w\right) \tag{25}$$

$$u^* \nabla \mathcal{P} = \mathcal{S}D - \frac{c_p}{c_v}\left(D - \frac{1}{H}\tilde{\partial} w\right) \tag{26}$$

$$u^* \nabla Q' = -\mathcal{N}D + \frac{1}{RT^*} u^* \nabla \phi_S \tag{27}$$

Structure equation can be obtained from system (23)–(27) by eliminating all variables except w. It is analogical to (14):

$$\nabla \left[-\frac{1}{c^2} \bar{u}_D \bar{u}_w {u^*}^2 \Delta + u^* \left(\bar{u}_w \Delta + \bar{u}_D \frac{\mathcal{L}}{H^2} \right) + N^2 \right] w = 0 \tag{28}$$

General solution of equation (28) can be written as a superposition of particular solutions having form:

$$w = \hat{w}\, \mathrm{e}^{ikx} \sigma^{i\nu - \frac{1}{2}} \tag{29}$$

$$k \in \mathbb{R} \qquad \hat{w}, \nu \in \mathbb{C}$$

Inserting (29) into structure equation (28) leads to dispersion formula analogical to (16):

$$\frac{\nu^2}{H^2} = \frac{N^2}{u^* \hat{u}_D} - k^2 \frac{\hat{u}_w}{\hat{u}_D}\left(1 - \frac{u^* \hat{u}_D}{c^2}\right) - \frac{1}{4H^2} \tag{30}$$

Determining complex amplitude $\hat{w}$ corresponding to monochromatic orographic forcing is straightforward in this case. It is sufficient to insert expressions $\phi_S = \hat{\phi}_S\, \mathrm{e}^{ikx}$ and $w = \hat{w}\, \mathrm{e}^{ikx} \sigma^{i\nu - \frac{1}{2}}$ into free slip BBC:

$$g w_S = u^* \nabla \phi_S$$

$$\Downarrow$$

$$g \hat{w} = iku^* \hat{\phi}_S \tag{31}$$

Having expression (31), it is possible to write down vertical velocity field $w(x, \sigma)$:

$$g w(x, \sigma) = iku^* \hat{\phi}_S\, \mathrm{e}^{ikx} \sigma^{i\nu - \frac{1}{2}} \tag{32}$$

Solution (32) is identical to (21). There is no diffusive chimney when vertical velocity w is used as prognostic variable.

B Reverted bubble test

Aim of this section is to show that when Boussinesq approximation is applicable, reverted bubble test should give the same results as direct test.

Dynamical equations describing irrotational adiabatic frictionless atmosphere composed of perfect gas are usually written in the form:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{RT}{p}\nabla p - \nabla\phi \tag{33}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\varkappa p\,\nabla\cdot\mathbf{v} \tag{34}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -(\varkappa - 1)T\,\nabla\cdot\mathbf{v} \tag{35}$$

$$\phi \equiv gz \qquad \varkappa \equiv \frac{c_p}{c_v}$$

Standard notations are used: $\mathbf{v}$ is 3D velocity with components (u, v, w), p is pressure, T is thermodynamical temperature, ϕ is geopotential, g is gravity acceleration, R is gas constant of dry air, c_p and c_v are specific heats of dry air at constant pressure and at constant volume.

Equations (33)–(35) can be rewritten into more suitable form using non-dimensional Exner function Π and potential temperature θ:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -c_p\theta\,\nabla\Pi - \nabla\phi \tag{36}$$

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} = -(\varkappa - 1)\Pi\,\nabla\cdot\mathbf{v} \tag{37}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0 \tag{38}$$

$$\Pi \equiv \left(\frac{p}{p_{00}}\right)^{\kappa} \qquad \theta \equiv T\left(\frac{p_{00}}{p}\right)^{\kappa}$$

$$\kappa \equiv \frac{R}{c_p} \qquad p_{00} \equiv 1000\,\mathrm{hPa}$$

Quantities Π and θ can be decomposed into background values and perturbations:

$$\Pi = \Pi_0 + \Pi' \qquad \theta = \theta_0 + \theta' \tag{39}$$

Background state is chosen resting, hydrostatically balanced and neutrally stratified (isoentropic). This gives:

$$\Pi_0(z) = \Pi_0(0) - \frac{gz}{c_p\theta_0} \qquad \theta_0 = const \tag{40}$$

Inserting (39) and (40) into system (36)–(38) with restriction to xz plane leads to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c_p(\theta_0 + \theta')\frac{\partial \Pi'}{\partial x} \tag{41}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -c_p(\theta_0 + \theta')\frac{\partial \Pi'}{\partial z} + g\frac{\theta'}{\theta_0} \tag{42}$$

$$\frac{\partial \Pi'}{\partial t} + u\frac{\partial \Pi'}{\partial x} + w\frac{\partial \Pi'}{\partial z} = \frac{gw}{c_p\theta_0} - (\varkappa - 1)(\Pi_0 + \Pi')\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \tag{43}$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \tag{44}$$

It should be mentioned here that equations (41)–(44) still describe the full 2D system, i.e. no simplifications were used during their derivation. At this point Boussinesq approximation can be introduced. It is based on two basic assumptions:

1. Perturbation θ' is small compared to θ_0 and can be neglected in equations (41), (42) except from buoyant term $g\dfrac{\theta'}{\theta_0}$.
2. Flow is close to incompressible. This requires two things: fluid velocity much smaller than speed of sound and vertical scale of motion small compared to density scale height.

Both these assumptions were fulfilled in bubble tests described in section 3, so the use of Boussinesq approximation should be justified. When it is applied to system (41)–(44), it becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c_p\theta_0\frac{\partial \Pi'}{\partial x} \tag{45}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -c_p\theta_0\frac{\partial \Pi'}{\partial z} + g\frac{\theta'}{\theta_0} \tag{46}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{47}$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \tag{48}$$

System (45)–(48) has interesting symmetry, responsible for identical behaviour of direct and reverted bubble test. It can be revealed using vertical mirroring operator M_z defined as:

$$(M_z f)(x, z, t) \equiv f(x, H - z, t)$$

Function f must be defined for $z \in [0, H]$. It can be shown easily that operator M_z is linear and has following properties:

$$\begin{aligned} M_z\partial_t &= \partial_t M_z \\ M_z\partial_x &= \partial_x M_z \\ M_z\partial_z &= -\partial_z M_z \\ M_z(f \cdot g) &= M_z f \cdot M_z g \end{aligned}$$

Using these properties it can be verified immediately that system (45)–(48) is invariant with respect to transformation:

$$\begin{pmatrix} u \\ w \\ \Pi' \\ \theta' \end{pmatrix} \mapsto \begin{pmatrix} M_z u \\ -M_z w \\ M_z \Pi' \\ -M_z \theta' \end{pmatrix}$$

This means that when fields u, w, Π', θ' are solution of the system (45)–(48), their vertical mirroring with change of sign for w and θ' produces another solution.

Remark:

For experiments described in section 3 initial state was resting ($u = 0$, $w = 0$) and vertically balanced. If it was horizontally balanced, initial perturbation Π' would be zero. Neverthless, Π' was very small initially, so the only quantity which was actually mirrored when preparing initial state for reverted test was perturbation θ'. This small inconsistency might be the reason why there is a slight difference visible when comparing figures 10 and 14. Another possible explanation is that this difference was caused by non-Boussinesq effects allowed by model dynamics.

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