1 Pseudo-prognostic TKE scheme

1.1 Vertical diffusion scheme in Aladin

Vertical diffusion scheme use in Aladin model is a Louis type scheme (K-closure) that explicitly resolves boundary layer. The fluxes are assumed proportional to gradients:

$$\overline{u'w'} = -K_m \frac{\partial u}{\partial z} \qquad \overline{v'w'} = -K_m \frac{\partial v}{\partial z} \qquad \overline{\theta' w'} = -K_h \frac{\partial \theta}{\partial z}$$

where the values for the turbulent transfer coefficients $K_{m,h,E}$ are diagnosed every timestep.

$$K_m = l_m^2 \left| \frac{\partial V}{\partial z} \right| F_m(R_i) \qquad K_h = l_m l_h \left| \frac{\partial V}{\partial z} \right| F_h(R_i)$$

The final tendency of the variable due to vertical diffusion is:

$$\left(\frac{\partial u}{\partial t} \right)_{turb} = \frac{\partial}{\partial z} K_m \frac{\partial u}{\partial z} \qquad \left(\frac{\partial v}{\partial t} \right)_{turb} = \frac{\partial}{\partial z} K_m \frac{\partial v}{\partial z} \qquad \left(\frac{\partial \theta}{\partial t} \right)_{turb} = \frac{\partial}{\partial z} K_h \frac{\partial \theta}{\partial z}$$

The scheme allows simple shallow convection parameterization through a modification of Richardson number Ri and antifibrillation sceme.

1.2 Full TKE equation

The full prognostic equation for turbulent kinetic energy (TKE) $E = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$ is

$$\frac{dE}{dt} = \underbrace{-\overline{u'w'}\frac{\partial u}{\partial z} - \overline{v'w'}\frac{\partial v}{\partial z}}_{\text{mech prod by wind shear}} - \underbrace{\frac{g}{\theta_0}\overline{\theta' w'}}_{\text{buoy prod/dest}} - \underbrace{\frac{\partial}{\partial z}\left(\overline{E'w'} + \frac{\overline{p'w'}}{\rho} \right)}_{\text{transport and diffusion}} - \underbrace{\epsilon}_{\text{dissipation}}$$

The pseudo prognostic equation for pTKE is

$$\frac{dE}{dt} = - \underbrace{\frac{\partial}{\partial z}\left(-K_E \frac{\partial E}{\partial z} \right)}_{\text{transport and diffusion}} + \underbrace{\frac{1}{\tau_E}(\tilde{E} - E)}_{\text{mech, buoy and dissipation}}$$

so $\overline{E'w'} + \frac{\overline{p'w'}}{\rho} = -K_E \frac{\partial E}{\partial z}$, $\tau_E = \frac{E}{\epsilon}$ and $\tilde{E}$ is a kind of a balance state for wind shear production and buoyancy dissipation.

1.3 pTKE procedure

- compute turbulent exchange coefficients from the model variables in the usual way (Louis)

$$\tilde{K}_m = l_m^2 \left| \frac{\partial V}{\partial z} \right| F_m(R_i) \qquad \tilde{K}_n = l_m^2 \left| \frac{\partial V}{\partial z} \right| \qquad \tilde{K}_h = l_m l_h \left| \frac{\partial V}{\partial z} \right| F_h(R_i)$$

- use them to compute

$$\tilde{K}_* = R_l \tilde{K}_n^{1-\gamma} \tilde{K}_m^{\gamma}$$

  where R_l ensures dominance of the diffussion process over the Newtonian relaxation one.

- use $\tilde{K}_*$ to compute

$$\tilde{E} = \left(\frac{\tilde{K}_*}{\nu l_m} \right)^2$$

- and

$$K_E = \frac{\tilde{K}_*}{\nu^2} \qquad \frac{1}{\tau_E} = \frac{\nu^2 \tilde{K}_*}{l_m^2} \qquad \text{1st timestep}$$

$$K_E = \frac{l_m}{\nu} \sqrt{E} \qquad \frac{1}{\tau_E} = \frac{\nu^3 \sqrt{E}}{l_m} \qquad \text{otherwise}$$

- and these are used to compute the tendency of TKE

$$\left(\frac{\partial E}{\partial t} \right)_{phy} = \frac{\partial}{\partial z} K_E \frac{\partial E}{\partial z} + \frac{1}{\tau_E} (\tilde{E} - E)$$

- the new value for TKE is used to compute

$$K_* = \nu l_m \sqrt{E}$$

- which is used to rescale the turbulent transfer coefficients

$$K_m = \frac{K_*}{\tilde{K}_*} \tilde{K}_m \qquad K_h = \frac{K_*}{\tilde{K}_*} \tilde{K}_h$$

2 General subgrid model from Redelsperger et al. 2001 (RMC)

2.1 The problem

Reducing the turbulent kinetic energy equation to a balance between dissipation ϵ and shear production

$$\epsilon = -\overline{u'w'}\frac{\partial u}{\partial z} - \overline{v'w'}\frac{\partial v}{\partial z}$$

using the common subgrid scale eddy viscosity approach

$$-\overline{u'w'} = K_m\frac{\partial u}{\partial z} \qquad -\overline{v'w'} = K_m\frac{\partial v}{\partial z} \qquad \text{where} \qquad K_m = C_K L\sqrt{E} \qquad \text{and} \qquad \epsilon = \frac{E^{\frac{3}{2}}}{\frac{L}{C_\epsilon}}$$

where L is the mesh size, $C_K = \frac{1}{\pi}\left(\frac{2}{2\alpha_3}\right)^{\frac{3}{2}}$ and $C_\epsilon = \pi\left(\frac{2}{2\alpha_3}\right)^{\frac{3}{2}}$ measurements give $\alpha_3 = 1.6$ what leads to $C_K = 0.0857$ and $C_\epsilon = 0.8445$.

Using all of the above equations, the subgrid turbulent kinetic energy is

$$\frac{E^{\frac{3}{2}}}{\frac{L}{C_\epsilon}} = C_K L\sqrt{E}\left(\frac{\partial u}{\partial z}\right)^2 + C_K L\sqrt{E}\left(\frac{\partial v}{\partial z}\right)^2$$

$$E = \frac{C_K}{C_\epsilon}L^2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]$$

and the Reynolds stresses are

$$\overline{u'w'}^2 + \overline{v'w'}^2 = \frac{C_K^3}{C_\epsilon}L^4\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^2$$

but from the similarty laws, we get

$$E = \alpha u_*^2 = \alpha\kappa^2 z^2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]$$

so we get $L = \kappa z$, $C_epsilon = \frac{1}{\alpha^{3/2}}$ and $C_K = \frac{1}{\sqrt{\alpha}}$ where $alpha = 3.75 - 5.47$ as suggested by published data.

2.2 Solution

In the general case (non-neutral)

$$-\overline{u'w'}\frac{\partial u}{\partial z} - \overline{v'w'}\frac{\partial v}{\partial z} + \frac{g}{\theta_0}\overline{\theta' w'} - \epsilon = 0$$

the bouyancy flux in

$$-\overline{\theta' w'} = K_h \frac{\partial \theta}{\partial z} \qquad K_h = C_H L_K \sqrt{E}\phi_3 \qquad \phi_3 = \frac{1}{1 + \frac{C_H}{C_\theta}\frac{g}{\theta_0}\frac{L_K L_\epsilon}{E}\frac{\partial \theta}{\partial z}}$$

The constants $C_H = \frac{1}{\pi}\frac{4}{3\beta_3}\left(\frac{2}{3\alpha_3}\right)^{\frac{1}{2}}$ and $2C_\epsilon = \pi\frac{4}{3\beta_3}\left(\frac{2}{3\alpha_3}\right)^{\frac{1}{2}}$ are computed from closure terms, $\beta_3 = 1.34$.

The turbulent kinetic energy, momentum and heat fluxes are:

$$E = \frac{C_K}{C_\epsilon}L^2\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right] f(Ri)$$

$$\overline{u'w'}^2 + \overline{v'w'}^2 = \frac{C_K^3}{C_\epsilon}L^4\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^2 f(Ri)$$

$$-\overline{\theta' w'} = C_H\frac{C_K^{\frac{1}{2}}}{C_\epsilon^{\frac{1}{2}}}L^2\frac{\partial \theta}{\partial z}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{1}{2}} \phi_3(Ri) f(Ri)^{\frac{1}{2}}$$

The similarity laws give

$$\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]^{\frac{1}{2}} = \frac{u_*}{\kappa z}\phi_m$$

$$\frac{\partial \theta}{\partial z} = \frac{T_*}{\kappa z}\phi_h$$

3 The stability dependency tuning of the pTKE scheme

The stability dependency tuning of the pTKE scheme is derived in the following way:

$$K_m = \nu l_m \phi_L \sqrt{E} \qquad \frac{1}{\tau_\epsilon} = \frac{\nu^3 \sqrt{E}}{l_m \psi_L}$$

where ϕ_L and ψ_L are functions of stability $\frac{K_m}{K_n}$, we will use them from RMC01:

$$\phi_L = \frac{1}{\phi_m^2 \phi_E^{\frac{1}{2}}} \qquad \psi_L = \frac{\phi_E^{\frac{3}{2}} \phi_m^2}{f}$$

the following scaling properties are used

$$\nu_* = \frac{\nu}{\psi_L^{\frac{1}{3}}} \qquad K_* = \frac{K_m}{\phi_L \psi_L^{\frac{1}{3}}}$$

We get

$$K_* = \frac{K_m}{\phi_L \psi_L^{\frac{1}{3}}} = \frac{\nu l_m \phi_L \sqrt{E}}{\phi_L \psi_L^{\frac{1}{3}}} = \frac{\nu_* \psi_L^{\frac{1}{3}} l_m \phi_L \sqrt{E}}{\phi_L \psi_L^{\frac{1}{3}}} = \nu_* l_m \sqrt{E}$$

Introducing the RMC01 functions for ϕ_L and ψ_L we get

$$K_* = \frac{K_m}{\frac{1}{\phi_m^2 \phi_E^{\frac{1}{2}}} \left(\frac{\phi_E^{\frac{3}{2}} \phi_m^2}{f} \right)^{\frac{1}{3}}} = \frac{K_m \phi_m^2 \phi_E^{\frac{1}{2}} f^{\frac{1}{3}}}{\phi_E^{\frac{1}{2}} \phi_m^{\frac{2}{3}}} = K_m \phi_m (\phi_m f)^{\frac{1}{3}}$$

Using the hypotheses that are strictly valid only around neutrality, $f \phi_m = 1$ and $\phi_m^2 = \frac{K_n}{K_m}$, we get

$$K_* = \sqrt{K_n K_m}$$

therefore, in order to keep this property, we should ensure that $\phi_m (\phi_m f)^{\frac{1}{3}} = \sqrt{\frac{K_n}{K_m}}$ for the whole range of stabilities.

The change of variables combined with the RMC01 functions also gives:

$$\nu_* = \frac{\nu}{\psi_L^{\frac{1}{3}}} = \frac{\nu}{\left(\frac{\phi_E^{\frac{3}{2}}\phi_m^2}{f}\right)^{\frac{1}{3}}} = \frac{\nu f^{\frac{1}{3}}}{\phi_E^{\frac{1}{2}}\phi_m^{\frac{2}{3}}} = \frac{\nu(f\phi_m)^{\frac{1}{3}}}{\phi_E^{\frac{1}{2}}\phi_m}$$

using the same hypotheses as for K_*, and itroducing the RMC01 functions for ϕ_E and ϕ_m we get

$$\nu_* = \frac{\nu}{\phi_E^{\frac{1}{2}}\phi_m} = \frac{\nu}{\left(\left[1+\frac{1}{\alpha}(-\frac{z}{L})^{\frac{2}{3}}+\frac{\beta}{\alpha\kappa^{\frac{2}{3}}}(-\frac{z_i}{L})^{\frac{2}{3}}\right]\frac{1}{\phi_m^2}\right)^{\frac{1}{2}}\phi_m} = \frac{\nu}{\left[1+\frac{1}{\alpha}(-\frac{z}{L})^{\frac{2}{3}}+\frac{\beta}{\alpha\kappa^{\frac{2}{3}}}(-\frac{z_i}{L})^{\frac{2}{3}}\right]^{\frac{1}{2}}}$$

ignoring the dependence on $\frac{z_i}{L}$ we get

$$\nu_* = \frac{\nu}{\left[1+\frac{1}{\alpha}(-\frac{z}{L})^{\frac{2}{3}}\right]^{\frac{1}{2}}}$$